doi:10.1006/jabr.2001.8722, available online at http://www.idealibrary.com on IDEAL®

Duality and Rational Modules in Hopf Algebras over Commutative Rings[1]

J. Y. Abuhlail

Institute of Mathematics, University of Düsseldorf, D-40225 Düsseldorf, Germany
E-mail: abuhlail@math.uni-duesseldorf.de

and

J. Gómez-Torrecillas and F. J. Lobillo

Departamento de Álgebra, Universidad de Granada, E-18071 Granada, Spain
E-mail: torrecil@ugr.es, jlobillo@ugr.es

Communicated by Susan Montgomery

Received March 28, 2000

Let A be an algebra over a commutative ring R. If R is noetherian and A° is pure in R^A, then the categories of rational left A-modules and right A°-comodules are isomorphic. In the Hopf algebra case, we can also strengthen the Blattner–Montgomery duality theorem. Finally, we give sufficient conditions to get the purity of A° in R^A. © 2001 Academic Press

Key Words: rational module; comodule; duality.

INTRODUCTION

It is well known that the theory of Hopf algebras over a field cannot be trivially passed to Hopf algebras over a commutative ring. For instance let us consider $\mathbb{Z}[x]$ as Hopf algebra and let $\mathfrak{a}$ be the Hopf ideal generated by $\langle 4, 2x\rangle$. Let H be the Hopf $\mathbb{Z}$-algebra $H = \mathbb{Z}[x]/\mathfrak{a}$. The finite dual is zero in this situation. However, $H \cong \mathbb{Z}_4[x]/\langle 2x\rangle$, so we can view H as a Hopf $\mathbb{Z}_4$-algebra. If I is a $\mathbb{Z}_4$-cofinite ideal of H then every element nonzero in

[1] The authors are grateful for the financial support by the Spanish–German exchange program *Acciones Integradas.*

0021-8693/01 $35.00

H/I has order 2, which implies every element in H° has order 2. In this situation H° is not pure in $\mathbb{Z}_4^H = \mathrm{Map}(H, \mathbb{Z}_4)$ as a $\mathbb{Z}_4$-module (since H° is not free) and a canonical $\mathbb{Z}_4$-coalgebra structure on H° cannot be expected.

A basic result from the theory of coalgebras over fields is that the comodules are essentially rational modules. Thus, given a (left) non-singular pairing of a coalgebra C and an algebra A (see [8]), the categories of rational left A-modules and right C-comodules are isomorphic. This applies in particular for the canonical pairings (C, C^*) and (A°, A) derived from a coalgebra C and an algebra A, respectively. An attempt to develop systematically the theory of rational modules associated to a pairing (C, A), where C is a coalgebra and A is an algebra over an arbitrary commutative ring R, is [4]. A corollary of the theory developed there is that if C is projective as an R-module, then the category of right C-comodules is isomorphic to the category of rational left modules over the convolution dual R-algebra $C^* = \mathrm{Hom}_R(C, R)$ (this result was independently obtained in [13] by means of a different approach). However, the results of [4] are proved in a framework which does not allow us to apply them directly to the pairing (A°, A), for a given R-algebra A. In fact, the first problem is to endow the finite dual A° with a comultiplication, which entails some serious technical difficulties at the very beginning due to the lack of exactness of the tensor product bifunctor $- \otimes_R -$. Nevertheless, it has been recently proved in [1, Theorem 2.8] that if R is noetherian and A° is pure in the R-module R^A of all maps from A to R, then A° is a coalgebra. We have observed that the notion of rational pairing introduced in [4] can be restated in order that the methods developed there can be applied to the pairing (A°, A) to prove that the category of right A°-comodules is isomorphic to the category of rational left A-modules. This applies, in particular, for any algebra A over a hereditary noetherian commutative ring. We explain our general theory of rational modules and comodules in Sections 2 and 3.

We apply our methods to strengthen the Blattner–Montgomery duality theorem for Hopf algebras over commutative rings. Let H be a Hopf algebra over the commutative ring R. When R is a field, the Blattner–Montgomery duality theorem says that if U is a Hopf subalgebra of H°, A is an H-module algebra such that the H-action is locally finite in a sense appropriate to the choice of U, H and U have bijective antipodes, and there is a certain right–left symmetry in the action of $H\#U$ on H, then

$$(A\#H)\#U \cong A \otimes (H\#U).$$

There are two proofs of this theorem in the literature. The first one appeared in [2, Theorem 2.1], and a new one, due to Blattner, appears in [5, Theorem 9.4.9]. Since, in this situation, the U-comodules are just the U-locally finite H-modules, it is easy to see that the two theorems are equivalent. In the case of a general commutative ring R, there is a similar theorem due to Van den Bergh (see [11]) when H is finitely generated and projective over R. A generalization of [2] for Hopf algebras over a Dedekind domain R was proven by Chen and Nichols, under the technical condition that U is R-closed in H° (see [3, Theorem 5]). This condition guarantees that every U-locally finite is rational. However, it is not evident that [3, Theorem 5] generalizes [5, Theorem 9.4.9].

We show that the ideas used in the proof of [5, Theorem 9.4.9], together with our results on rational modules and comodules, can be combined to get a duality theorem for Hopf algebras over a noetherian commutative ring R which generalized both [5, Theorem 9.49] and [3, Theorem 5] and, hence, [2, Theorem 2.1].

In Section 4 we introduce a class of R-algebras $\mathrm{P}_{\ell}\mathrm{Alg}_R$ (in case R is noetherian) which satisfies the property that $A^{\circ} \subset R^A$ is an R-pure submodule for each $A \in \mathrm{P}_{\ell}\mathrm{Alg}_R$ and, hence, the canonical pairing $(A^{\circ}, A, \langle -, - \rangle)$ is a rational pairing. We give several examples of such algebras, among them the polynomial algebra $R[x_1, \dots, x_n]$ and the algebra of Laurent polynomials $R[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$.

1. PRELIMINARIES AND BASIC NOTIONS

In this paper R is a commutative ring with unit. Let A be an associative R-algebra with unit. The category of all left A-modules is denoted by ${}_A\mathcal{M}$. As usual, the notation $X \in \mathcal{C}$, where $\mathcal{C}$ is a category, means X is an object of $\mathcal{C}$. By $\otimes$ we denote the tensor product $\otimes_R$ unless otherwise explicitly stated. Moreover, if $\pi \in S_n$ (the symmetric group on n symbols) then τ_π is the canonical isomorphism

$$\tau_\pi : M_1 \otimes \cdots \otimes M_n \longrightarrow M_{\pi(1)} \otimes \cdots \otimes M_{\pi(n)}.$$

Let M, X be R-modules. If N is an R-submodule of M then N is called X-pure if $N \otimes X \subseteq M \otimes X$. The inclusion $N \subseteq M$ is called pure if N is X-pure for all R-modules X. Unless otherwise stated, pure, projective, and flat mean pure, projective, and flat in ${}_R\mathcal{M}$.

Let $\mathcal{A}$ be a Grothendieck category. A *preradical* for $\mathcal{A}$ is a subfunctor of the identity endofunctor $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. We follow [9] for categorical basic notions.

DEFINITION 1.1. An R-coalgebra is an R-module C together with two homomorphisms of R-modules

$$\Delta : C \to C \otimes C \text{ (comultiplication)} \qquad \text{and} \qquad \epsilon : C \to R \text{ (counit)}$$

such that

$$(id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$$

and

$$(id_C \otimes \epsilon) \circ \Delta = (\epsilon \otimes id_C) \circ \Delta = id_C.$$

DEFINITION 1.2. A right C-comodule is an R-module together with an R-homomorphism

$$\rho_M : M \to M \otimes C$$

such that

$$(id_M \otimes \Delta) \circ \rho_M = (\rho_M \otimes id_C) \circ \rho_M$$

and

$$(id_M \otimes \epsilon) \circ \rho_M = id_M.$$

Let M, N be right C-comodules. A homomorphism of R-modules $f : M \to N$ is said to be a comodule morphism (or C-colinear) if $\rho_N \circ f = (f \otimes id) \circ \rho_M$. By $\mathrm{Hom}^C(M, N)$ we denote the R-module of all colinear maps from M to N. The right C-comodules with the C-colinear maps between them constitute an additive category denoted by $\mathcal{M}^C$. In case that C is a flat R-module, $\mathcal{M}^C$ is a Grothendieck category (see [13, Corollary 3.15]).

For basic notions on coalgebras and comodules over commutative rings we refer to [4, 13].

2. RATIONAL MODULES

In [4], the theory of rational modules is developed under the assumption of the existence of the there-named rational pairing. The main example exhibited was (C^*, C) with C an R-projective coalgebra. Our aim is to deal with the finite dual of an R-algebra, so in this section we provide a weaker definition of rational pairing (in order to cover the finite dual example) which also implies the results on rational modules developed in [4].

2.1. *Rational Systems.* Let A, P be R-modules and let

$$\langle -, - \rangle : P \times A \to R$$

be a bilinear form. For every R-module M, define the R-linear map

$$\begin{aligned} \alpha_M : M \otimes P &\longrightarrow \mathrm{Hom}_R(A, M) \\ m \otimes p &\longmapsto [a \mapsto m\langle p, a\rangle]. \end{aligned}$$

PROPOSITION 2.1. *In the previous situation the following statements are equivalent*:

(1) α_M *is injective*

(2) *If* $\sum m_i \otimes p_i \in M \otimes P$, *then* $\sum m_i \otimes p_i = 0$ *if and only if for every* $a \in A$, $\sum m_i \langle p_i, a\rangle = 0$.

Proof. Note that for each R-module M,

$$\ker(\alpha_M) = \left\{ \sum m_i \otimes p_i \mid \sum \langle p_i, -\rangle m_i = 0 \right\}. \qquad \blacksquare$$

DEFINITION 2.2. The three-tuple $(P, A, \langle -, -\rangle)$ is a *rational system* if α_M is injective for every R-module M.

Remark 2.3. By [4, Proposition 2.3] and Proposition 2.1, a rational system as defined in [4, Definition 2.1] is a rational system in the present setting.

Remark 2.4. Let$(P, A, \langle -, -\rangle)$ be a rational system. Let M be an R-module and let N be an R-submodule of M. Consider the following commutative diagram

$$\begin{tikzcd} N \otimes P \arrow[r, hook, "\alpha_N"] \arrow[d, "i_N \otimes id_P"'] & \mathrm{Hom}_R(A, N) \arrow[d, hook, "i"] \\ M \otimes P \arrow[r, hook, "\alpha_M"'] & \mathrm{Hom}_R(A, M) \end{tikzcd}$$

Note that α_N is injective since the three-tuple $(P, A, \langle -, -\rangle)$ is a rational system, and $i : \mathrm{Hom}_R(A, N) \to \mathrm{Hom}_R(A, M)$, $g \mapsto i_N \circ g$ is injective. Hence $i_N \otimes id_P$ is injective. Since M was arbitrary in ${}_R\mathcal{M}$ we conclude that P should be flat as an R-module.

The following proposition replaces [4, Proposition 2.2] in order to show that the canonical comodule structure over a rational module is pseudocoassociative

PROPOSITION 2.5. *If* $(P, A, \langle -, -\rangle)$ *and* $(Q, B, \langle -, -\rangle)$ *are rational systems, then the induced pairing*

$$[-, -] : P \otimes Q \times A \otimes B \to R$$

defined by

$$\left[\sum p_i \otimes q_i, \sum a_j \otimes b_j\right] = \sum \langle p_i, a_j\rangle \langle q_i, b_j\rangle$$

is a rational system.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} M \otimes P \otimes Q & \xrightarrow{\quad \beta_M \quad} & \mathrm{Hom}_R(A \otimes B, M) \\ \Big\downarrow{\scriptstyle \alpha_{M\otimes P}} & & \Big\downarrow{\scriptstyle \eta} \\ \mathrm{Hom}_R(B, M \otimes P) & \xrightarrow{(\alpha_M)_*} & \mathrm{Hom}_R(B, \mathrm{Hom}_R(A, M)) \end{array}$$

where η is the adjunction isomorphism, β_M is the mapping analogous to α with respect to the pairing $[-,-]$, and $(\alpha_M)_*$ is the homomorphism induced by α_M. This last morphism is monic and $\alpha_{M\otimes P}$ is monic. Therefore, β_M is a monomorphism. ∎

For each set S, let R^S denote the R-module of all maps from S to R. If α_R is injective, then P is isomorphic to a submodule of $A^* = \mathrm{Hom}_R(A, R) \subseteq R^A$. We identify the R-module P with its image in R^A, so every $p \in P$ is identified with the R-linear map $\langle p, -\rangle : A \to R$.

DEFINITION 2.6. We say P is *mock-projective* (relative to A and $\langle -,-\rangle$) if α_R is injective and for every $p_1, \dots, p_n \in P$ there are $a_1, \dots, a_m \in A$ and $g_1, \dots, g_m \in R^A$ such that for every $i = 1, \dots, n$, $p_i = \sum \langle p_i, a_l \rangle g_l$.

Proposition 2.1 can be improved under the assumption that P is mock-projective:

PROPOSITION 2.7. *Assume P is mock-projective. If $M \in {}_R\mathcal{M}$, then $P \subseteq R^A$ is M-pure if and only if α_M is injective. Therefore, $(P, A, \langle -,-\rangle)$ is a rational system if and only if $P \subseteq R^A$ is pure.*

Proof. Assume $P \subseteq R^A$ is M-pure and let $\sum m_i \otimes p_i \in M \otimes P$. Assume $\sum \langle p_i, a \rangle m_i = 0$ for all $a \in A$. By Definition 2.6 there are $a_1, \dots, a_m \in A$ and $g_1, \dots, g_m \in R^A$ such that

$$p_i = \sum \langle p_i, a_l \rangle g_l, \qquad \text{for each } i = 1, \dots, n.$$

As $\sum m_i \otimes p_i \in M \otimes R^A$ we have

$$\begin{aligned} \sum m_i \otimes p_i &= \sum m_i \otimes \sum \langle p_i, a_l \rangle g_l = \sum m_i \langle p_i, a_l \rangle \otimes g_l \\ &= \sum \Big(\sum m_i \langle p_i, a_l \rangle \Big) \otimes g_l = \sum 0 \otimes g_l = 0, \end{aligned} \tag{1}$$

and by purity $\sum m_i \otimes p_i = 0$ as an element in $M \otimes P$. By Proposition 2.1, α_M is injective.

Assume now α_M is injective. The diagram

$$\begin{array}{ccc} M \otimes P & \xrightarrow{\alpha_M} & \mathrm{Hom}_R(A, M) \\ {\scriptstyle id\otimes i}\Big\downarrow & & \Big\downarrow{\scriptstyle i} \\ M \otimes R^A & \xrightarrow[\beta_M]{\qquad} & M^A \end{array}$$

is commutative where i denotes inclusion and $\beta_M(m \otimes f)(a) = mf(a)$. Since $i \circ \alpha_M$ is injective, we get $id \otimes i$ is injective and $P \subseteq R^A$ is M-pure. ∎

2.2. *Rational Pairings.* Let (C, Δ, ϵ) be an R-coalgebra and let (A, m, u) be an R-algebra.

DEFINITION 2.8. A *rational pairing* is a rational system $(C, A, \langle -, - \rangle)$ where C is an R-coalgebra, A is an R-algebra, and the map $\varphi : A \to C^*$ given by $\varphi(a)(c) = \langle c, a \rangle$ is a homomorphism of R-algebras. This is equivalent to requiring

$$\langle -, - \rangle \circ (id_C \otimes m) = (\langle -, - \rangle \otimes \langle -, - \rangle) \circ \tau_{(23)} \circ (\Delta \otimes id_A^{\otimes 2})$$

and

$$\epsilon = \langle -, - \rangle \circ (id_C \otimes u) = \langle -, 1 \rangle. \tag{2}$$

Now we can parallel the definition and properties of rational modules in [4]. The proofs are formally the same as in [4]. We include some of them for convenience of the reader.

DEFINITION 2.9. Let $T = (C, A, \langle -, - \rangle)$ be a rational pairing. An element m in a left A-module M is called *rational* (with respect to the pairing T) if there exist finite subsets $\{m_i\} \subseteq M$ and $\{c_i\} \subseteq C$ such that $am = \sum m_i \langle c_i, a \rangle$ for every $a \in A$. The subset $\{(m_i, c_i)\} \subseteq M \times C$ is called a *rational set of parameters for m* (with respect to the pairing T). The subset $\mathrm{Rat}^T(M)$ of M consisting of all rational elements of M is clearly an R-submodule of M. A left A-module is called *rational* (with respect to the pairing T) if $M = \mathrm{Rat}^T(M)$. The full subcategory of ${}_A\mathcal{M}$ whose objects are all the rational (with respect to the pairing T) left A-modules will be denoted by $\mathrm{Rat}^T({}_A\mathcal{M})$. We use the notation Rat instead of Rat^T when the rational pairing T is clear from the context.

Remark 2.10. As a consequence of Proposition 2.1, if $\{(m_i, c_i)\} \subseteq M \times C$ is a rational set of parameters for m and $\sum m_i \otimes c_i = \sum n_j \otimes d_j$, then $\{(n_j, d_j)\} \subseteq M \times C$ is a rational set of parameters for m. In fact a rational set of parameters for m can be viewed as a representative of an element $\sum m_i \otimes c_i \in M \otimes C$.

Following [12], we will denote by $\sigma[{}_AC]$ the full subcategory of ${}_A\mathcal{M}$ consisting of all the left A-modules subgenerated by ${}_AC$. This means that a left A-module belongs to $\sigma[{}_AC]$ if and only if it is isomorphic to a submodule of a factor module of a direct sum of copies of ${}_AC$.

THEOREM 2.11. *Let $T = (C, A, \langle -, - \rangle)$ be a rational pairing. Then*

(1) $\mathrm{Rat} = \mathrm{Rat}^T : {}_A\mathcal{M} \to {}_A\mathcal{M}$ *is a left exact preradical.*
(2) *The categories* $\mathrm{Rat}({}_A\mathcal{M})$ *and* $\mathcal{M}^C$ *are isomorphic.*
(3) $\mathrm{Rat}({}_A\mathcal{M}) = \sigma[{}_AC]$.

Proof. The proofs of these facts are formally the same as [4, Propositions 2.9 and 2.10, Theorems 3.12 and 3.13], using Propositions 2.1 and 2.5 instead of [4, Propositions 2.3 and 2.2]. ∎

The isomorphism given in Theorem 2.11 is defined in terms of sets of rational parameters: if $M \in \mathrm{Rat}({}_A\mathcal{M})$ then the structure of right C comodule is $\omega_M(m) = \sum m_i \otimes c_i$ where $\{(m_i, c_i)\}$ is a set of rational parameters; if $(M, \delta_M) \in \mathcal{M}^C$ and $\delta_M(m) = \sum m_i \otimes c_i$, then $\{(m_i, c_i)\}$ is a set of rational parameters for m. See [4, Propositions 3.5 and 3.11] for details.

Sweedler's Σ-notation can be introduced in terms of sets of rational parameters. Let $T = (C, A\langle -, -\rangle)$ be a rational pairing. We have the injective R-linear map $\alpha_R : C \to A^*$ defined by

$$\alpha_R : C \longrightarrow A^*$$
$$c \longmapsto [a \mapsto \langle c, a\rangle].$$

Let us regard A^* as a left A-module via $(a\lambda)(b) = \lambda(ba)$ and let us identify C with $\alpha_R(C)$. As in [4, Proposition 3.2], $C = \mathrm{Rat}({}_A A^*)$. Note that the set of rational parameters for $c \in C$ is given by $\Delta(c)$. If $\{(c_1, c_2)\}_{(c)} = \{(c_1, c_2)\} \subseteq C \times C$ represents a set of rational parameters of $c \in C$ then the comultiplication can be represented as

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = \sum c_1 \otimes c_2.$$

Note that (2) means

$$\langle c, ab\rangle = \sum \langle c_1, a\rangle \langle c_2, b\rangle$$

in Σ-notation. Analogously, let $(M, \delta_M) \in \mathcal{M}^C$. The set of rational parameters for $m \in M$ is given by $\delta_M(m)$. We are going to use Sweedler's Σ-notation on C-comodules, i.e., $\delta_M(m) = \sum_{(m)} m_0 \otimes m_1$ where $\{(m_0, m_1)\}_{(m)} \subseteq M \times C$ represents an arbitrary set of rational parameters for $m \in M$.

2.3. *The Finite Dual Coalgebra A°. In this subsection, the commutative ring R is assumed to be noetherian.* Let A be an R-algebra. Recall that the canonical structure of an A-bimodule on A^* is given by

$$(af)(b) = f(ba) \quad \text{and} \quad (fa)(b) = f(ab) \text{ for } f \in A^* \text{ and } a, b \in A. \tag{3}$$

Let $A^\circ = \{f \in A^* \mid Af$ is finitely generated as an R-module$\}$. Then by [1, Proposition 2.6],

$$\begin{aligned} A^\circ &= \{f \in A^* \mid Af \text{ is finitely generated as an } R\text{-module}\} \\ &= \{f \in A^* \mid fA \text{ is finitely generated as an } R\text{-module}\} \\ &= \{f \in A^* \mid \ker f \text{ contains an } R\text{-cofinite ideal of } A\} \\ &= \{f \in A^* \mid \ker f \text{ contains an } R\text{-cofinite left ideal of } A\} \\ &= \{f \in A^* \mid \ker f \text{ contains an } R\text{-cofinite right ideal of } A\}. \end{aligned} \tag{4}$$

By [1, 2.3], A° is an A-subbimodule of A^*.

Remark 2.12. If A is finitely generated and projective in ${}_R\mathcal{M}$ then $A^\circ = A^*$ is pure in R^A. In this case $(A \otimes A)^* \simeq A^* \otimes A^*$.

By [1, Theorem 2.8], A° is an R-coalgebra whenever A° is a pure submodule of R^A. In this section we will prove that $(A^\circ, A, \langle -, - \rangle)$ is a rational pairing. So we are going to describe the right A°-comodules as rational left A-modules. This applies in particular when R is hereditary.

The next lemma is used to prove the rationality of the three-tuple $(A^\circ, A, \langle -, - \rangle)$.

LEMMA 2.13. *Let S be a set and let $f_1, \dots, f_n \in R^S$. Then there exist $s_1, \dots, s_m \in S$ and $g_1, \dots, g_m \in R^S$ such that*

$$f_i = \sum f_i(s_l) g_l, \qquad \textit{for each } i = 1, \dots, n.$$

In particular, if A, P are R-modules and $\langle -, - \rangle : P \times A \to R$ a bilinear form such that α_R is injective, then P is mock-projective.

Proof. Define

$$\begin{aligned} \underline{f} : S &\longrightarrow R^n \\ s &\longmapsto (f_1(s), \dots, f_n(s)), \end{aligned}$$

and consider the R-submodule $M \subseteq R^n$ generated by the elements $\underline{f}(s)$, $s \in S$. Since R is noetherian, M is finitely generated, and there are $s_1, \dots, s_m \in S$ for which $M = \sum R\underline{f}(s_l)$. For any $s \in S$ we have the set

$$X(s) = \{(r_1, \dots, r_m) \in R^m \mid \underline{f}(s) = \sum r_l \underline{f}(s_l)\} \neq \emptyset.$$

For each $s \in S$, choose $r(s) \in X(s)$. This gives a map $r : S \to R^m$. Now, it is clear that there are maps $g_1, \dots, g_m \in R^S$ such that $r = (g_1, \dots, g_m)$. Finally, $\underline{f}(s) = \sum g_l(s) \underline{f}(s_l)$. But this is an equality in R^n, whence, for each $i = 1, \dots, n$ we obtain $f_i(s) = \sum g_l(s) f_i(s_l)$, and hence $f_i = \sum f_i(s_l) g_l$. ∎

Remark 2.14. Assume α_R to be injective. Let M be an R-module. It follows directly from Lemma 2.13 and Proposition 2.7 that $P \subseteq R^A$ is M-pure if and only if α_M is injective.

PROPOSITION 2.15. *Let A be an R-algebra and assume A° is pure in R^A. Then*

(1) *A° is an R-coalgebra. If in addition A is a bialgebra (resp. Hopf algebra) then A° is a bialgebra (resp. Hopf algebra).*

(2) *Let B be an R-algebra such that B° is pure in R^B. For every morphism of R-algebras $\varphi : A \to B$ we have*

$$\varphi^*(B^\circ) \subseteq A^\circ;$$

moreover $\varphi^\circ := \varphi^*_{|B^\circ}$ is an R-coalgebra morphism.

(3) *Let $C \subseteq A^\circ$ be a subcoalgebra. Consider the bilinear form*

$$\begin{aligned} \langle -, - \rangle : C \times A &\longrightarrow R \\ (f, a) &\longmapsto \langle f, a \rangle = f(a). \end{aligned}$$

Then $(C, A, \langle -, - \rangle)$ is a rational pairing.

(4) *If $(C, A, \langle -, - \rangle)$ is a rational pairing then C is an R-subcoalgebra of A°.*

Proof. (1) By [1, Theorem 2.8], A° is a coalgebra. The rest of the first statement is similar to the argument in [5, 9.1.3] due to the fact that over noetherian rings, submodules of finitely generated modules are finitely generated.

(2) Let $f \in B^\circ$ and assume $I \subseteq B$ to be a cofinite left ideal contained in $\ker f$. Since R is noetherian, it is easy to check that $\varphi^{-1}(I) \subseteq A$ is a left cofinite ideal contained in $\ker(f \circ \varphi) = \ker(\varphi^*(f))$. A diagram chase shows that φ° is an R-coalgebra map and the second statement is proved.

(3) Equation (2) is clearly satisfied. By [13, 3.3], C is pure in A°, so Proposition 2.7 and Lemma 2.13 give injectivity of α_M for each R-module M.

(4) Since α_R is injective C can be viewed as an R-submodule of A^*. Let us see that $C \subseteq A^\circ$. If $c \in C$ and $\Delta(c) = \sum c_1 \otimes c_2$, then $ac = \sum c_1 \langle c_2, a \rangle$ by Eqs. (2) and (3). It follows that Ac is finitely generated by $\{c_1\}_{(c)}$ as an R-module for every $c \in C$. Moreover, Eq. (2) easily implies that the comultiplication and the counit on C are induced from A°. ∎

COROLLARY 2.16. *Let A be an R-algebra and assume A° is pure in R^A. Consider the bilinear form*

$$\begin{aligned} [-, -] : A^\circ \times A^{\circ *} &\longrightarrow R \\ (f, \lambda) &\longmapsto [f, \lambda] = \lambda(f) \end{aligned}$$

for all $f \in A^\circ$ and $\lambda \in A^{\circ *}$. Then $(A^\circ, A^{\circ *}, [-, -])$ is a rational pairing.

Proof. Assume $\sum [f_i, \lambda] m_i = 0$ for all $\lambda \in A^{\circ *}$. Let $a \in A$ and consider

$$\begin{aligned} \langle -, a \rangle : A^\circ &\longrightarrow R \\ f &\longmapsto \langle f, a \rangle = f(a) \end{aligned}$$

for all $f \in A^\circ$. Then $\sum [f_i, \langle -, a \rangle] m_i = 0$ and so $\sum \langle f_i, a \rangle m_i = 0$ for all $a \in A$, which implies $\sum m_i \otimes f_i = 0$, since the pairing $(A^\circ, A, \langle -, - \rangle)$ is a rational pairing by Proposition 2.15. ∎

As a consequence of Theorem 2.11, Propositions 2.15 and 2.1, and Corollary 2.16 we have:

THEOREM 2.17. *Let A be an R-algebra such that A° is pure in R^A. Let $\varphi : A \to A^{\circ *}$ be the canonical morphism and let $\varphi_* : {}_{A^{\circ *}}\mathcal{M} \to {}_A\mathcal{M}$ be the restriction of scalars functor. Then*

(1) *The functors $(-)^{A^\circ} : \mathrm{Rat}({}_A\mathcal{M}) \to \mathcal{M}^{A^\circ}$ and $(-)^{A^\circ} : \mathrm{Rat}({}_{A^{\circ *}}\mathcal{M}) \to \mathcal{M}^{A^\circ}$ are isomorphisms of categories.*

(2) $\mathrm{Rat}({}_A\mathcal{M}) = \sigma[{}_A A^\circ]$ *and* $\mathrm{Rat}({}_{A^{\circ *}}\mathcal{M}) = \sigma[{}_{A^{\circ *}} A^\circ]$.

(3) *The following diagram of functors is commutative*

$$\begin{tikzcd}
{}_{A^{\circ *}}\mathcal{M} \arrow[rr, "\varphi_*"] \arrow[d, "\mathrm{Rat}^{T'}"'] & & {}_A\mathcal{M} \arrow[d, "\mathrm{Rat}^{T}"] \\
\mathrm{Rat}^{T'}({}_{A^{\circ *}}\mathcal{M}) \arrow[dr, "\simeq", "(-)^{A^\circ}"'] & & \mathrm{Rat}^{T}({}_A\mathcal{M}) \arrow[dl, "\simeq"', "(-)^{A^\circ}"] \\
& \mathcal{M}^{A^\circ} &
\end{tikzcd}$$

where $T = (A^\circ, A\langle -, -\rangle)$ and $T' = (A^\circ, A^{\circ *}, [-, -])$ are the canonical pairings.

3. AN APPLICATION: BLATTNER–MONTGOMERY DUALITY

We are going to prove a Blattner–Montgomery like theorem (see [5, Theorem 9.4.9]) when R *is any commutative noetherian ring* and $(H, m, u, \Delta, \epsilon, S)$ is a Hopf algebra such that H° is pure in R^H (this condition holds if H is R-projective and H° is pure in H^*). When R is a Dedekind domain, we obtain as a corollary the version given in [3].

We are going to recall some definitions and notations. A left H-module algebra is an R-algebra (A, m_A, u_A) such that A is a left H-module and m_A, u_A are H-module maps. This means in terms of Sweedler's notation,

$$h(ab) = \sum (h_1 a)(h_2 b) \qquad \text{and} \qquad h1_A = \epsilon(h)1_A.$$

Analogously A is a right H-comodule algebra if A is a right H-comodule (via $\rho : A \to A \otimes H$) and m_A, u_A are H-comodule maps, i.e.,

$$\rho(ab) = \sum a_0 b_0 \otimes a_1 b_1 \qquad \text{and} \qquad \rho(1_A) = 1_A \otimes 1_H.$$

Let A be a left H-module algebra where the H-module action is denoted by $w_A : H \otimes A \to A$. The following composition of maps

$$
\begin{tikzcd}
(A \otimes H) \otimes (A \otimes H) \arrow[r, "id\otimes\Delta\otimes id^{\otimes 2}"] \arrow[dd, dashed, "m_{A\#H}"'] & A \otimes H \otimes H \otimes A \otimes H \arrow[d, "\tau_{(34)}"] \\
 & A \otimes H \otimes A \otimes H \otimes H \arrow[d, "id\otimes w_A\otimes id^{\otimes 2}"] \\
A \otimes H & A \otimes A \otimes H \otimes H \arrow[l, "m_A\otimes m"]
\end{tikzcd}
$$

provides a structure of an associative R-algebra on $A \otimes H$. This algebra is called the smash product of A and H, and it is denoted by $A\#H$. In Sweedler's notation, the multiplication can be viewed as

$$(a\#h)(b\#k) = \sum a(h_1 b)\#h_2 k,$$

where $a\#h = a \otimes h$. Since H° is pure in R^H, by Proposition 2.15 we have $(H^\circ, \Delta^\circ, \epsilon^\circ, m^\circ, u^\circ, S^\circ)$ is a Hopf algebra. The left (and right) action of H on H° described in (3) makes H° a left (and right) H-module algebra (see [5, Example 4.1.10]). In order to make the notation consistent with the literature we denote the left (resp. right) action of H on H° by $\rightharpoonup$ (resp. $\leftharpoonup$). Let U be a Hopf subalgebra of H° (by definition $U \subseteq H^\circ$ should be pure; see [13, 3.3]). Then U is also a left H-module algebra. The action can be described as

$$H \otimes U \xrightarrow{id\otimes m^\circ} H \otimes U \otimes U \xrightarrow{\tau_{(123)}} U \otimes H \otimes U \xrightarrow{\langle -, - \rangle \otimes id} U,$$

and in Sweedler's notation,

$$h \rightharpoonup f = \sum f_1 \langle f_2, h \rangle$$

which allows the construction of $U\#H$.

Analogously H is a left (resp. right) U-module algebra via

$$U \otimes H \xrightarrow{id\otimes\Delta} U \otimes H \otimes H \xrightarrow{\tau_{(23)}} U \otimes H \otimes H \xrightarrow{\langle -, - \rangle \otimes id} H$$

$$(\text{resp. } H \otimes U \xrightarrow{\Delta\otimes id} H \otimes H \otimes U \xrightarrow{\tau_{(123)}} U \otimes H \otimes H \xrightarrow{\langle -, - \rangle \otimes id} H).$$

This action is denoted by $\rightharpoonup$ (resp. $\leftharpoonup$), and Sweedler's notation means

$$f \rightharpoonup h = \sum h_1 \langle f, h_2 \rangle \qquad (\text{resp. } h \leftharpoonup f = \sum \langle f, h_1 \rangle h_2)$$

and we can construct $H\#U$.

These actions and constructions are analogous to the ones over a field. See [5, 1.6.5, 1.6.6, 4.1.10] for details. Following [5, Definition 9.4.1] we have the maps

$$\begin{aligned} \lambda : H\#U &\longrightarrow \mathrm{End}_R(H) \\ h\#f &\longmapsto [k \mapsto h(f \rightharpoonup k)] \\ \rho : U\#H &\longrightarrow \mathrm{End}_R(H) \\ f\#h &\longmapsto [k \mapsto (k \leftharpoonup f)h]. \end{aligned}$$

LEMMA 3.1. *λ is an algebra morphism and ρ is an anti-algebra morphism. If also H has bijective antipode, then λ and ρ are injective.*

Proof. Following [5, Lemma 9.4.2], we consider λ, as the argument for ρ is similar. Straightforward computations show that λ is an algebra morphism. To see the injectivity we define $\lambda' : H\#U \to \mathrm{End}_R(H)$ and $\psi : \mathrm{End}_R(H) \to \mathrm{End}_R(H)$ as

$$\begin{aligned} \lambda'(h\#f)(k) &= \langle f, k\rangle h \\ \psi(\sigma) &= (\sigma \otimes \overline{S}) \circ \tau \circ \Delta, \end{aligned}$$

where $\overline{S}$ is the composition inverse of S. We can see that $\lambda' = \psi \circ \lambda$ as in [5, Lemma 9.4.2]. Moreover, $(U, H, \langle -, -\rangle)$ is a rational pairing by Proposition 2.15, so λ' is injective. ∎

We say that U satisfies the RL-condition with respect to H if $\rho(U\#1) \subseteq \lambda(H\#U)$.

Let (A, ρ_A) be a right U-comodule algebra. Then A is a left H-module algebra with action

$$H \otimes A \xrightarrow{id\otimes\rho_A} H \otimes A \otimes U \xrightarrow{\tau_{(132)}} A \otimes U \otimes H \xrightarrow{id\otimes\langle -,-\rangle} A, \tag{5}$$

or in Sweedler's notation,

$$ha = \sum a_0 \langle a_1, h\rangle.$$

THEOREM 3.2. *Let H be a Hopf algebra such that H° is pure in R^H, and let U be a Hopf subalgebra of H°. Assume that both H and U have bijective antipodes and U satisfies the* RL*-condition with respect to H. Let A be a right U-comodule algebra. Let U act on $A\#H$ by acting trivially on A and via $\rightharpoonup$ on H. Then*

$$(A\#H)\#U \simeq A \otimes (H\#U).$$

Proof. The computations in [5, Theorem 9.4.9 and Lemma 9.4.10] remain valid here once we have proved Lemma 3.1. ∎

Remark 3.3. Let R be a Dedekind domain and assume that A is an U-locally finite left H-module algebra and that U is R-closed in H° in the sense of [3]. By [3, Lemma 4], A is a rational left H-module which implies, by Theorem 2.17, that A is a right U-comodule algebra. Therefore, [3, Theorem 5] follows as a corollary of Theorem 3.2.

Remark 3.4. If H is cocommutative then U satisfies the RL-condition (see [5, 9.4.7 Example]), so examples in Subsections 4.2 and 4.3 and Example 4.7 satisfy the RL-condition. So let G be a group such that $R[G]^\circ$ is pure in $R^{R[G]}$ (if G is either finite or R is hereditary, this condition is satisfied), and let A be an R-algebra such that G acts as automorphisms on A. Then we have

$$(A\#R[G])\#R[G]^\circ \cong A \otimes (R[G]\#R[G]^\circ).$$

4. EXAMPLES

In this section R is assumed to be noetherian. We are going to consider a class of R-algebras for which A° is pure in R^A (and hence A° has a structure of an R-coalgebra). For every R-algebra A let $\mathscr{L}_{\mathrm{cof}}$ be the linear topology on A whose basic neighborhoods of 0 are the R-cofinite left ideals, i.e.,

$$\mathscr{L}_{\mathrm{cof}} = \{I \leq {}_A A \mid A/I \text{ is finitely generated as an } R\text{-module}\}.$$

4.1. *The Category* $\mathrm{P}_\ell\mathrm{Alg}_R$.

DEFINITION 4.1 (Property $\mathbf{P}_\ell$). An R-algebra A has property $\mathbf{P}_\ell$ in case the set

$$\mathscr{P}_{\mathrm{cof}} = \{I \leq {}_A A \mid A/I \text{ is finitely generated and projective as an } R\text{-module}\}$$

is a basis for $\mathscr{L}_{\mathrm{cof}}$; i.e., for every left cofinite ideal I of A, there exists a left ideal $I_0 \subseteq I$, with A/I_0 finitely generated and projective as an R-module.

We denote by $\mathrm{P}_\ell\mathrm{Alg}_R$ the full subcategory of Alg_R whose objects are all R-algebras which have property $\mathbf{P}_\ell$.

PROPOSITION 4.2. *If $A \in \mathrm{P}_\ell\mathrm{Alg}_R$, then $(A^\circ, A, \langle -, - \rangle)$ is a rational system.*

Proof. Let $M \in {}_R\mathscr{M}$ and let $\sum m_i \otimes f_i \in M \otimes A^\circ$. Assume $\sum m_i \langle f_i, a\rangle = 0$ for every $a \in A$. Notice that for each i, $f_i \in (A/I_i)^*$ for some cofinite ideal I_i of A. Put $J = \bigcap_i I_i$. Then J is cofinite. Since $A \in \mathrm{P}_\ell\mathrm{Alg}_R$ there exists some ideal $J_0 \subseteq J$ such that A/J_0 is finitely generated and projective as an R-module (and so $(A/J_0)^{**} \simeq A/J_0$). Let $\{a_\lambda + J_0, \phi_\lambda\}_\Lambda$ be a finite dual basis for $(A/J_0)^*$. Since $f_i \in (A/J_0)^*$ for all i, we get

$$\begin{aligned}\sum m_i \otimes f_i &= \sum m_i \otimes \sum \langle f_i, a_\lambda + J_0\rangle \phi_\lambda\\ &= \sum \left(\sum \langle f_i, a_\lambda + J_0\rangle m_i\right) \otimes \phi_\lambda\\ &= \sum 0 \otimes \phi_\lambda = 0 \qquad (\text{notice that } f_i(J_0) = 0).\end{aligned}$$

Hence $(A^\circ, A, \langle -, -\rangle)$ is a rational system by Proposition 2.1. ∎

COROLLARY 4.3. *If* $A \in \mathrm{P}_\ell\mathrm{Alg}_R$ *then*

(1) A° *is an* R*-coalgebra. If in addition* A *is a bialgebra* (*resp. Hopf algebra*) *then* A° *is a bialgebra* (*resp. Hopf algebra*).

(2) $(A^\circ, A, \langle -, -\rangle)$ *is a rational pairing.*

Proof. It follows directly from Propositions 2.7, 2.15, and 4.2. ∎

Remark 4.4. By (4), the proof of Proposition 4.2 remains true if we replace left ideals in property $\mathbf{P}_\ell$ by right or two sided ones. So we can speak of property $\mathbf{P_r}$ or property $\mathbf{P}$.

Remark 4.5. If $A \in \mathrm{P}_\ell\mathrm{Alg}_R$, then

$$\begin{aligned}A^{\circ *} = \mathrm{Hom}_R(A^\circ, R) &= \mathrm{Hom}_R\Big(\varinjlim_{I \in \mathscr{L}_{\mathrm{cof}}} (A/I)^*, R\Big)\\ &\simeq \mathrm{Hom}_R\Big(\varprojlim_{I \in \mathscr{P}_{\mathrm{cof}}} (A/I)^*, R\Big)\\ &\simeq \varprojlim_{I \in \mathscr{P}_{\mathrm{cof}}} (A/I)^{**} \simeq \varprojlim_{I \in \mathscr{P}_{\mathrm{cof}}} A/I\\ &\simeq \varprojlim_{I \in \mathscr{L}_{\mathrm{cof}}} A/I = \widehat{A},\end{aligned}$$

which means that $A^{\circ *} \simeq \widehat{A}$, the completion of A with respect to the cofinite topology.

PROPOSITION 4.6. *Let* A *be in* $\mathrm{P}_\ell\mathrm{Alg}_R$ *and let* B *be an* R*-algebra extension of* A *such that* B *is finitely generated and projective in* $\mathscr{M}_A$. *Then* B *belongs to* $\mathrm{P}_\ell\mathrm{Alg}_R$.

Proof. Let $J \leq B$ be a cofinite left ideal. Then $J \cap A \leq A$ is a cofinite left ideal because R is noetherian. Since A belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$, there exists $I_0 \subseteq J \cap A$ such that A/I_0 is finitely generated and projective in ${}_R\mathscr{M}$. By the natural isomorphism

$$\mathrm{Hom}_R\left(B \otimes_A \frac{A}{i_0}, -\right) \cong \mathrm{Hom}_A\left(B, \mathrm{Hom}_R\left(\frac{A}{I_0}, -\right)\right).$$

$B \otimes_A (A/I_0)$ is finitely generated and projective in ${}_R\mathcal{M}$. Since $BI_0 \subseteq J$ and $B \otimes_A (A/I_0) \cong B/BI_0$ we get B is in $\mathrm{P}_\ell\mathrm{Alg}_R$. ∎

EXAMPLE 4.7. Let G be a group. An R-algebra is called G-graded if for every $\sigma \in G$ there exists an R-submodule $A_\sigma \subseteq A$ such that $A = \bigoplus_{\sigma\in G} A_\sigma$ and $A_\sigma A_\tau \subseteq A_{\sigma\tau}$. If in addition $A_\sigma A_\tau = A_{\sigma\tau}$, A is called strongly graded. Let G be finite with neutral element e and let A be a strongly G-graded R-algebra. By [6, I.3.3 Corollary] it is clear that A is finitely generated and projective as right A_e-module, so if A_e is in $\mathrm{P}_\ell\mathrm{Alg}_R$ then A also belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$. In particular, if A is in $\mathrm{P}_\ell\mathrm{Alg}_R$ and G is a finite group, then a crossed product $A * G$ also belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$. Crossed products cover the following cases: if $A \in \mathrm{P}_\ell\mathrm{Alg}_R$ then $A[G]$, $A^t[G]$, $AG \in \mathrm{P}_\ell\mathrm{Alg}_R$ where $A[G]$ is the group algebra, $A^t[G]$ is the twisted group algebra, and AG is the skew group algebra. See [7] for an introduction on crossed products.

Our aim is the proof of Theorem 4.10, which was shown in [10, Lemma 6.0.1] for algebras over fields and in [3] for algebras over Dedekind domains. However, we need some technical statements.

LEMMA 4.8. *Let M and N be two R-modules and consider submodules $M' \subseteq M$ and $N' \subseteq N$. Assume M' to be N-pure and N' to be M-pure (this is in particular valid if M and N are flat in ${}_R\mathcal{M}$). Then*

$$M/M' \otimes N/N' \simeq (M \otimes N)/(M' \otimes N + M \otimes N').$$

Proof. By purity $M' \otimes N$ and $M \otimes N'$ are R-submodules of $M \otimes N$. Since the diagram

$$\begin{tikzcd} M \otimes N \arrow[r, two heads] \arrow[d, two heads] & M/M' \otimes N \arrow[d, two heads] \\ M \otimes N/N' \arrow[r, two heads] & M/M' \otimes N/N' \end{tikzcd}$$

is a pushout diagram, the result follows. ∎

PROPOSITION 4.9. *Let A, B be algebras in $\mathrm{P}_\ell\mathrm{Alg}_R$.*

(1) *If $K \leq A \otimes B$ is a cofinite left ideal then there exist $I_0 \leq A$ and $J_0 \leq B$ such that A/I_0 and B/J_0 are finitely generated and projective in ${}_R\mathcal{M}$, and so that $I_0 \otimes B + A \otimes J_0 \subseteq K$.*

(2) *The R-algebra $A \otimes B$ belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$.*

Proof. Consider the canonical maps

$$\alpha : A \longrightarrow A \otimes B$$
$$a \longmapsto a \otimes 1$$

and

$$\beta : B \longrightarrow A \otimes B$$
$$b \longmapsto 1 \otimes b.$$

Put $I = \alpha^{-1}(K)$ and $J = \beta^{-1}(K)$. Since R is noetherian, I and J are cofinite left ideals of A and B, respectively. Since $A, B \in \mathrm{P}_\ell\mathrm{Alg}_R$ there exist $I_0 \subseteq I$ and $J_0 \subseteq J$ such that A/I_0 and B/J_0 are finitely generated and projective in ${}_R\mathcal{M}$. Let $K_0 = I_0 \otimes B + A \otimes J_0$. Since $I_0 \leq A$ and $J_0 \leq B$ are pure submodules we have $K_0 \subseteq K$ as desired.

By Lemma 4.8,

$$\frac{A \otimes B}{K_0} \simeq \frac{A}{I_0} \otimes \frac{B}{J_0},$$

hence $(A \otimes B)/K_0$ is finitely generated and projective and $A \otimes B$ is in $\mathrm{P}_\ell\mathrm{Alg}_R$. ∎

THEOREM 4.10. *Let A, B be in $\mathrm{P}_\ell\mathrm{Alg}_R$. Then there is a canonical isomorphism $A^\circ \otimes B^\circ \simeq (A \otimes B)^\circ$.*

Proof. Since A, B are in $\mathrm{P}_\ell\mathrm{Alg}_R$, A° is pure in R^A and B° is pure in R^B by Propositions 4.2 and 2.7. So $A^\circ \otimes B^\circ \subseteq R^A \otimes R^B$. Let π be the morphism

$$\pi : R^A \otimes R^B \longmapsto R^{A \times B}$$
$$f \otimes g \longmapsto [(a, b) \mapsto f(a)g(b)].$$

By [1, Proposition 1.2] this map is injective, so the statement will be clear once we have seen $\pi(A^\circ \otimes B^\circ) = (A \otimes B)^\circ$. So let $f \otimes g \in A^\circ \otimes B^\circ$ and let $I \subseteq A$ and $J \subseteq B$ be left ideals contained in $\ker f$ and $\ker g$, respectively, and such that A/I, B/J are finitely generated and projective (they exist because A, B belong to $\mathrm{P}_\ell\mathrm{Alg}_R$). Since $I \subseteq A$ and $J \subseteq B$ are pure, by Lemma 4.8, $I \otimes B + A \otimes J \subseteq A \otimes B$ is a cofinite left ideal, which is contained in $\ker(\pi(f \otimes g))$. As $\pi(f \otimes g)$ is bilinear it is clear that $\pi(A^\circ \otimes B^\circ) \subseteq (A \otimes B)^\circ$.

Let $h \in (A \otimes B)^\circ$ and assume $K \subseteq A \otimes B$ to be a cofinite left ideal contained in $\ker h$. By Proposition 4.9 there exist left ideals $I_0 \leq A$ and $J_0 \leq B$ such that A/I_0 and B/J_0 are finitely generated and projective in ${}_R\mathcal{M}$ and so that $I_0 \otimes B + A \otimes J_0 \subseteq K$. By Lemma 4.8 there is an epimorphism

$$\frac{A}{I_0} \otimes \frac{B}{J_0} \to \frac{A \otimes B}{K} \to 0$$

which induces a monomorphism

$$0 \to \left(\frac{A \otimes B}{K}\right)^* \to \left(\frac{A}{I_0} \otimes \frac{B}{J_0}\right)^* \simeq \left(\frac{A}{I_0}\right)^* \otimes \left(\frac{B}{J_0}\right)^* \subseteq A^\circ \otimes B^\circ.$$

So there exist elements $f_1, \ldots, f_n \in (A/I_0)^* \subseteq A^\circ$ and $g_1, \ldots, g_n \in (B/J_0)^* \subseteq B^\circ$ such that $\pi(\sum f_i \otimes g_i) = h$. This completes the proof. ∎

We finish with some examples.

4.2. *The R-Bialgebra* $R[x_1, \dots, x_n]^\circ$. By [1, Proposition 3.1], every cofinite ideal $I \leq R[x]$ contains a monic polynomial $f(x)$. Put $I_0 = (f(x)) \subseteq I$. Then $R[x]/I_0$ is finitely generated and projective (in fact free). Hence $R[x]$ is in $\mathrm{P}_\ell\mathrm{Alg}_R$ and so $R[x]^\circ$ is an R-coalgebra by Corollary 4.3. Moreover, $R[x_1, \dots, x_n]$ belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$ by Proposition 4.9. There are two canonical bialgebra structures on $R[x_1, \dots, x_n]$. The first one comes from the semigroup algebra structure of $R[x_1, \dots, x_n]$ (i.e., every x_i is a group-like element), and the second one appears when we see $R[x_1, \dots, x_n]$ as the enveloping algebra of an abelian Lie algebra (i.e., every x_i is a primitive element). The latter one is a Hopf algebra structure. By Corollary 4.3, $R[x_1, \dots, x_n]^\circ$ is a bialgebra (resp. Hopf algebra).

It follows from Proposition 4.9 that if A belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$ then $A[x_1, \dots, x_n]$ is in $\mathrm{P}_\ell\mathrm{Alg}_R$.

4.3. *The Hopf R-Algebra of Laurent Polynomials.*

DEFINITION 4.11. A monic polynomial $q(x) \in R[x]$ is called reversible if $q(0)$ is a unit in R. An ideal $I \subseteq R[x, x^{-1}]$ is called reversible if it contains a reversible polynomial $q(x)$.

LEMMA 4.12. *Let $q(x) \in R[x]$ be a reversible polynomial. Then*

$$R[x]/(q(x)) \simeq R[x, x^{-1}]/(q(x)).$$

Proof. Let $q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a reversible polynomial (i.e., a_0 in a unit in R). Notice

$$R[x, x^{-1}]/(q(x)) \simeq R[x, y]/(xy - 1, q(x)).$$

Put $I = (xy - 1, q(x))$ and consider the R-linear map

$$\begin{aligned} \Psi : R[x] &\longrightarrow R[x, y]/I \\ x &\longmapsto x + I. \end{aligned}$$

Clearly Ψ is an R-algebra homomorphism and $\ker(\Psi) = I \cap R[x]$. Moreover, $I \cap R[x] = R[x]q(x)$. Clearly Ψ is surjective if and only if $y + I \in \mathrm{im}(\Psi)$. Notice

$$\begin{aligned} yq(x) - x^{n-1}(xy - 1) &= a_0 y + a_1 yx + \cdots + a_{n-1} yx^{n-1} + x^{n-1} \\ &= a_0 y + a_1 + a_2 x + \cdots + a_{n-1} x^{n-2} + x^{n-1} \quad \mathrm{mod}\,(I). \end{aligned}$$

So

$$y = -a_0^{-1}[x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1] \qquad \mathrm{mod}\,(I).$$

Hence $y \in \mathrm{im}(\Psi)$ and we conclude that Ψ is surjective. ∎

PROPOSITION 4.13. (1) *Let $I \subseteq R[x, x^{-1}]$ be a reversible ideal. Then $R[x, x^{-1}]/I$ is finitely generated as an R-module.*

(2) *Let R be noetherian. Assume $R[x, x^{-1}]/I$ to be finitely generated as an R-module. Then I is a reversible ideal.*

Proof. (1) Let $I \subseteq R[x, x^{-1}]$ be a reversible ideal. Then I contains a reversible polynomial $q(x)$. By Lemma 4.12, $R[x, x^{-1}]/(q(x)) \simeq R[x]/(q(x))$ which implies, by [1, Proposition 3.1], that $R[x, x^{-1}]/(q(x))$ is finitely generated as an R-module. Therefore, $R[x, x^{-1}]/I$ is finitely generated as an R-module.

(2) Since $R[x]/(R[x] \cap I)$ embeds in the finitely generated R-module $R[x, x^{-1}]/I$, we get that $R[x]/(R[x] \cap I)$ is finitely generated as an R-module. By [1], there exists a monic polynomial $f_1(x) = a_0 + a_1 x + \cdots + x^n \in I \cap R[x]$. We know $R[x, x^{-1}]$ is a Hopf R-algebra with antipode

$$\begin{aligned} S : R[x, x^{-1}] &\longrightarrow R[x, x^{-1}] \\ x &\longmapsto x^{-1}. \end{aligned}$$

Since S is bijective, $R[x, x^{-1}]/I \simeq R[x, x^{-1}]/S(I)$ as R-modules. So there exists a monic $f_2(x) = b_0 + \cdots + b_{m-1}x^{m-1} + x^m \in S(I) \cap R[x]$. Hence we have that $q(x) = x^m(f_1(x) + S(f_2(x))) \in I$. An easy computation shows that

$$q(x) = 1 + b_{m-1}x + \cdots + (b_0 + a_0)x^m + \cdots a_{n-1}x^{n+m-1} + x^{n+m}$$

and I contains the reversible polynomial $q(x)$. By Lemma 4.12

$$R[x, x^{-1}]/(q(x)) \simeq R[x]/(q(x))$$

and so is finitely generated and projective (in fact free) as an R-module. ∎

THEOREM 4.14. *Let R be noetherian and let A be in $\mathrm{P}_\ell\mathrm{Alg}_R$. Then $A[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ belongs to $\mathrm{P}_\ell\mathrm{Alg}_R$. In particular $R[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]^\circ$ is a Hopf algebra.*

Proof. By Proposition 4.13 and Lemma 4.12 it is easy to see that $R[x, x^{-1}]$ is in $\mathrm{P}_\ell\mathrm{Alg}_R$, so the first statement follows from Proposition 4.9. Since $R[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ is a group algebra, the last assertion follows from Corollary 4.3. ∎

REFERENCES

1. J. Y. Abuhlail, J. Gómez-Torrecillas, and R. Wisbauer, Dual coalgebras of algebras over commutative rings, *J. Pure Appl. Algebra* **153** (2000), 107–120.
2. R. J. Blattner and S. Montgomery, A duality theorem for Hopf module algebras, *J. Algebra* **95** (1985), 153–172.
3. C.-Y. Chen and W. D. Nichols, A duality theorem for Hopf module algebras over Dedekind rings, *Comm. Algebra* **18** (1990), 3209–3221.
4. J. Gómez Torrecillas, Coalgebras and comodules over a commutative ring, *Rev. Roumaine Math. Pures Appl.* **43** (1998), 591–603.
5. S. Montgomery, "Hopf Algebras and Their Actions on Rings," CBMS, Vol. 82, Amer. Math. Soc., Providence, 1993.
6. C. Năstăsescu and F. Van Oystaeyen, "Graded Ring Theory," North-Holland, Amsterdam, 1982.
7. D. S. Passman, "Infinite Crossed Products," Pure and Applied Mathematics, Vol. 135, Academic Press, San Diego, 1989.
8. D. E. Radford, Coreflexive coalgebras, *J. Algebra* **26** (1973), 512–535.
9. B. Stenström, "Rings of Quotients," Springer-Verlag, Berlin, 1975.
10. M. E. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.
11. M. Van den Bergh, A duality theorem for Hopf algebras, *in* "Methods in Ring Theory," NATO ASI, Vol. 129, Reidel, Dordrecht, 1984.
12. R. Wisbauer, "Foundations of Module and Ring Theory," Algebra, Logic and Appl., Vol. 3, Gordon & Breach, Philadelphia, 1991.
13. R. Wisbauer, Semiperfect coalgebras over rings, *in* "Proceedings, ICAC'97, Hongkong, 1997."